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Accuracy of the lattice Boltzmann method based on analytical solutions

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In this paper, a simple method is proposed to obtain steady analytical solutions for the lattice Boltzmann method. Based on such analytical results, it is demonstrated how the accuracy of the lattice Boltzmann method can depend on the relative orientation of the lattice and the flow field. It is also demonstrated that the method can be useful to obtain a general class of analytical solutions for the lattice Boltzmann method. Finally, a simple relation is given between the compressibility error and the velocity field.

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I. INTRODUCTION

Over the last few years, the lattice Boltzmann method (LBM) became a viable and useful method in the field of computational fluid dynamics. Due to some of its nice properties such as easy implementation and parallelization, it has been applied successfully in a wide range of hydrodynamic problems (for a review, see Refs. [1,2]). However, the accuracy of the LBM is still subject to debate. Using multiscale expansion, it has been shown that at low Mach numbers, the LBM solves the Navier-Stokes equations with second-order accuracy both in space and time [3]. Nevertheless, several numerical experiments have shown only first-order accuracy (see Ref. [4], and references therein). Now, it is generally accepted that the boundary conditions adopted from the lattice gas automaton method (e.g., bounce-back method for nonslip velocity) can reduce the accuracy of the LBM. Furthermore, this conclusion has been supported by analytical calculations. Indeed, when Noble *et al.* developed a new nonslip boundary condition for the LBM, they found that when using this boundary condition at the walls of a horizontal channel, the steady-state solution gave the Poiseuille profile up to the machine accuracy [5]. This observation suggested that a set of distribution functions has to exist, which is the exact representation of the Poiseulle flow (independent of the lattice spacing). Zhou *et al.* determined these distribution functions analytically, using the special properties of the horizontal Poiseuille flow [6]. They assumed, for instance, that the solution is symmetric, and independent of time and x. Based on this analytical solution, they also demonstrated that the bounce-back boundary condition can reduce the accuracy of the LBM to first order. However, the investigation focused only on the case where the lattice orientation was parallel to the walls of the channel (see Fig. 1, top left). Consequently, the analytical solutions obtained in this way could not answer the question how the bounce-back boundary condition performs in general, e.g., for inclined walls. To obtain *a priori* information about boundary conditions for such situations, one has to carry out the derivation of analytical solutions (if they exist at all) for curved walls. Unfortunately, there are only a few known analytical solutions for the Navier-Stokes equations where the geometrical domain is bounded by curved walls. Since for the horizontal Poiseuille flow we have an exact solution, it seems natural to look for an analytical solution for its rotated version; that is, rotating the geometrical domain on the lattice or, equivalently, rotating the lattice on the geometrical domain, one can look for analytical solutions for the inclined Poiseuille flow (see Fig. 1, right). Since for the rotated channel one of the properties used in Ref. [6] will be destroyed, namely, the x independence, one could not follow the steps given in Ref. [6].

The aims of this paper are threefold. First, using a simple idea it will be shown, how one can test whether an analytical steady-state solution of the Navier-Stokes equations can be obtained exactly (up to machine accuracy) in the framework of the LBM.

Second, using this idea the analytical solution of the rotated Poiseuille problem will be tested and will show how the accuracy of the LBM depends on the orientation of the lattice to the flow field for this specific case. We also analyze the accuracy of the analytical solution of a more complex

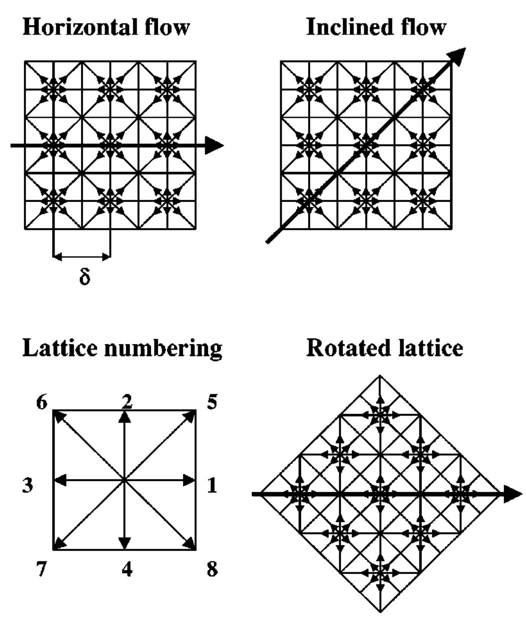

FIG. 1. Flow directions in horizontal and inclined channels ($-\pi/4$). This inclined channel corresponds to the horizontal flow in a rotated lattice.

*Permanent address: KFKI Atomic Energy Research Institute, Simulator Development Department, H-1525 Budapest, Hungary; electronic address: gah@sunserv.kfki.hu problem, namely, the Jeffery-Hamel flow, which has a close relationship with the Poiseuille problem.

Finally, we will give some preliminary results of an analysis, which focuses on the general class of analytical solutions of the LBM. It will be shown that for a specific class of problems [when $u_x = u_x(y)$ and $u_y = 0$], a simple condition can be obtained. It will be pointed out that for satisfying this condition the first-order solution in y gives the Couette flow, the second-order provides the Poiseulle flow, and we give a different analytical third-order solution, too. We point out that this class of flows is always free of compressibility error. For a more general class [when $u_x = u_x(x,y)$ and $u_y = u_y(x,y)$], a relation between the compressibility error and the velocity field is given.

II. THE LATTICE BOLTZMANN METHOD

In this paper, one of the most popular two-dimensional LBM models is used: D2Q9, which uses nine-velocities, but the idea to test analytical solutions can be immediately applicable to any other lattice [even for three-dimensional (3D) lattices].

The lattice Boltzmann equation using the Bhatnagar-Gross-Krook [7,8] collision operator is given by

$$f_i(\mathbf{r}+\delta\mathbf{e}_i, t+\delta) - f_i(\mathbf{r},t) = -\frac{1}{\tau}[f_i(\mathbf{r},t) - f_i^{eq}(\mathbf{r},t)], \quad (1)$$

where $f_i(\mathbf{r},t)$ is the particle distribution function, $\mathbf{e}_i$ is the lattice velocity vector, τ is the relaxation time that controls the rate of approach to the equilibrium distribution function $f_i^{eq}(\mathbf{r},t)$, and δ is the lattice spacing.

For a D2Q9 model, the lattice vectors (see Fig. 1, bottom left) take the form $\mathbf{e}_i = [\cos(\pi(i-1)/2), \sin(\pi(i-1)/2)]$ for $i = 1,2,3,4$ and $\mathbf{e}_i = \sqrt{2}[\cos(\pi(i-9/2)/2), \sin(\pi(i-9/2)/2)]$ for $i = 5,6,7,8$, and the equilibrium distributions can be given by

$$f_0^{eq} = w_0\rho\left(1 - \frac{3}{2}\mathbf{u}\cdot\mathbf{u}\right),$$
$$f_i^{eq} = w_i\rho\left[1 + 3(\mathbf{e}_i\cdot\mathbf{u}) + \frac{9}{2}(\mathbf{e}_i\cdot\mathbf{u})^2 - \frac{3}{2}\mathbf{u}\cdot\mathbf{u}\right], \quad (2)$$

where the lattice weights are $w_0 = \frac{4}{9}$, $w_i = \frac{1}{9}$ for $i = 1,2,3,4$ and $w_i = \frac{1}{36}$ for $i = 5,6,7,8$.

The equilibrium distributions are derived by the low-Mach number approximation of the Maxwell-Boltzmann distribution. A systematic and mathematically rigorous derivation of the coefficients can be found in Ref. [9].

The macroscopic quantities; hydrodynamic velocity and density are calculated by taking the following moments of the distribution functions:

$$\rho = \sum_i f_i, \quad \rho\mathbf{u} = \sum_i f_i\mathbf{e}_i. \quad (3)$$

By using the Chapman-Enskog multiscale expansion, it can be shown that the density and the velocities satisfy the Navier-Stokes equations in the low-Mach number limit (see the derivation, e.g., in Ref. [3]).

III. THE ROTATED POISEUILLE FLOW

Using the same model, Zou *et al.* [6] determined the form of the distribution functions that exactly satisfy the Poiseuille flow in a horizontal channel with unit width:

$$u_x = u_0(1-y^2), \quad u_y = 0, \quad \frac{\partial p}{\partial x} = -2\rho\nu u_0, \quad \frac{\partial p}{\partial y} = 0. \quad (4)$$

Their derivation was based on the special properties of this flow field (e.g., independency of t and x) and their results are valid in the overall parameter domain of τ, δ, u_0.

However, there is a more straightforward way to test analytical solutions in the LBM framework by choosing $\tau = 1$. Although this selection strictly limits the validation of any analytical solution, we can use this idea for more general problems where the flow field does not have such nice properties as those of the Poiseuille flow.

Indeed, choosing $\tau = 1$, the distribution functions at a given lattice site $\mathbf{r}$ and time $t+\delta$ will be determined by the equilibrium distributions of the neighboring sites based on the analytical solutions,

$$f_i(\mathbf{r}+\delta\mathbf{e}_i, t+\delta) = f_i^{eq}(\mathbf{r},t). \quad (5)$$

If the solution is steady, we should obtain the analytical solution at $\mathbf{r}$ by simply taking the corresponding moments of the distribution functions using Eq. (3). In other words, if one sums up the equilibrium distribution functions of the neighboring lattice sites of $\mathbf{r}$ (which correspond to an analytical solution) and adds the rest distribution of $\mathbf{r}$ to this sum, the result should be the analytical density at $\mathbf{r}$. For the horizontal Poiseuille flow, it is easy to check that the above leads to the same distribution functions that were obtained in Ref. [6]. (a substitution of $\tau = 1$ is understood for a direct comparison). Actually, the idea can be applied to test any steady state solution of the Navier-Stokes equation in the LBM framework, but an existing analytical solution would be validated in this way only for $\tau = 1$ or, from another point of view, for a fixed viscosity.

Using the idea above, one can derive analytical solution for the Poiseuille flow in an inclined channel. For simplicity, we first consider the problem where the angle of the channel is $-\pi/4$ (see Fig. 1, top right). A transformation of the analytical solution of the horizontal Poiseuille flow gives

$$u_x(x,y) = u_y(x,y) = \frac{\sqrt{2}}{2}u_0\left(1 - \frac{\sqrt{2}}{2}(y-x)^2\right). \quad (6)$$

Looking for the solution at $\mathbf{r} = (x,y)$ and substituting the analytical solutions into the equilibrium distributions at the neighboring sites, one obtains the following functions:

$$f_0^{eq}(x,y) = \frac{4\rho}{9}\left\{1 - \frac{3}{2}[u_x^2(x,y) + u_y^2(x,y)]\right\},$$

$$f_1^{eq}(x^-,y)=\frac{\rho}{9}[1+3u_x(x^-,y)+3u_x^2(x^-,y)],$$

$$f_2^{eq}(x,y^-)=\frac{\rho}{9}[1+3u_y(x,y^-)+3u_y^2(x,y^-)], \tag{7}$$

$$f_3^{eq}(x^+,y)=\frac{\rho}{9}[1-3u_x(x^+,y)+3u_y^2(x^+,y)],$$

$$f_4^{eq}(x,y^+)=\frac{\rho}{9}[1-3u_y(x,y^+)+3u_y^2(x,y^+)],$$

$$f_5^{eq}(x^-,y^-)=\frac{\rho}{36}\bigg\{1+3[u_x(x^-,y^-)+u_y(x^-,y^-)] + \frac{9}{2}[u_x(x^-,y^-)+u_y(x^-,y^-)]^2 - \frac{3}{2}[u_x^2(x^-,y^-)+u_y^2(x^-,y^-)]\bigg\},$$

$$f_6^{eq}(x^-,y^+)=\frac{\rho}{36}\bigg\{1+3[-u_x(x^-,y^+)+u_y(x^-,y^+)] + \frac{9}{2}[-u_x(x^-,y^+)+u_y(x^-,y^+)]^2 - \frac{3}{2}[u_x^2(x^-,y^+)+u_y^2(x^-,y^+)]\bigg\},$$

$$f_7^{eq}(x^+,y^+)=\frac{\rho}{36}\bigg\{1+3[-u_x(x^+,y^+)+u_y(x^+,y^+)] + \frac{9}{2}[-u_x(x^+,y^+)-u_y(x^+,y^+)]^2 - \frac{3}{2}[u_x^2(x^+,y^+)+u_y^2(x^+,y^+)]\bigg\},$$

$$f_8^{eq}(x^+,y^-)=\frac{\rho}{36}\bigg\{1+3[u_x(x^+,y^-)-u_y(x^+,y^-)] + \frac{9}{2}[u_x(x^+,y^-)-u_y(x^+,y^-)]^2 - \frac{3}{2}[u_x^2(x^+,y^-)+u_y^2(x^+,y^-)]\bigg\}, \tag{8}$$

where for conciseness, the superscript + and − have been introduced for displacements, e.g.,

$$f_7^{eq}(x^+,y^+)\equiv f_7^{eq}(x+\delta,y+\delta). \tag{9}$$

These functions will form the distributions at **r** and time $t+\delta$. The macroscopic quantities at **r** can be obtained from Eq. (3) using the equilibrium functions above. If the macroscopic quantities obtained are the analytical solutions at site **r**, then these distribution functions will form exact representations of the inclined Poiseuille flow at **r**. Summing up the distribution functions above, one obtains the following density:

$$\rho(1-Eu_0^2\delta^4), \tag{10}$$

where $E=1/4$.

This result clearly demonstrates that the distribution functions obtained do not satisfy the analytical solution since the density now depends on the lattice space δ. The error in the density is of the fourth order in space and the magnitude of the error is directly proportional to M^2, where M is the Mach number $M=u/c_s$ and c_s is the sound speed ($c_s^2=1/3$ for D2Q9). The rotation of the geometrical domain reduces the accuracy of the LBM. Note that we have not taken into account any driving force for the Poiseuille flow. However, this does not alter the fact that the solution of the flow is now lattice space dependent. Obviously, the introduction of a body force must not influence the density of the flow field. One can also obtain the same result by rotating the lattice by $-\pi/4$ for the original horizontal Poiseuille problem (Fig. 1, bottom right).

To investigate the effect of rotation on the velocity field, a body force is introduced to the calculation which transfers the same momentum to the fluid as the pressure gradient:

$$\frac{\partial p}{\partial x}=\frac{\partial p}{\partial y}=G=-\sqrt{2}\rho\nu u_0. \tag{11}$$

The implementation of the body force in the LBM is somewhat arbitrary (see for a review, Ref. [10]). However, to keep the analysis as near as possible to that given in Ref. [6], the body force

$$\frac{w_i}{c_s^2}\delta G_\alpha e_{i\alpha} \tag{12}$$

appears as an additional term on the right-hand side (RHS) of Eq. (1).

The velocity error for the horizontal Poiseuille profile with a $-\pi/4$ rotation of the lattice (see Fig. 1, bottom right) is given by

$$u_{x,err}=0, \quad u_{y,err}=u_0^2 y\delta^3. \tag{13}$$

The accuracy of the velocity is reduced to third order in space.

The error coefficient E changes with the angle of the rotation as follows:

$$E=-\cos^2(\alpha)+\cos^4(\alpha), \tag{14}$$

where α is the rotation angle (i.e., it has a maximum at $-\pi/4$, and it becomes zero at 0 and $-\pi/2$).

Note that the error has a maximum when there is a parallel lattice link (the cross diagonal of the lattice) with the orientation of the wall, and the error is zero only when the flow is parallel with the main links (as was pointed out in Ref. [6]). In spite of the fact that the solution found is not analytical in the LBM, one could use the error function obtained to *a priori* study boundary conditions in the very same way as it was done in Ref. [6]. Indeed, taking into account the effect of boundary conditions to the equilibrium distributions, one could determine the accuracy change of the solution and consequently the accuracy of the boundary condition itself.

IV. THE JEFFERY-HAMEL PROBLEM

We also performed a similar analysis for the Jeffery-Hamel problem. This is a radial flow caused by a sink or source at the origin and bounded by solid walls at $\theta = \pm\alpha$. Since this flow is a radial one, it could provide more information about the accuracy of the LBM as the flow and lattice directions are varied. For general cases, the solution of this problem is given by implicit elliptic integrals, but for creeping flow one can obtain the following simple explicit form [11]:

$$g(\eta) = 1 + \frac{1}{2}\csc^2(\alpha)[(\sin(\pi/2) - 2\alpha\eta) - 1], \qquad (15)$$

where $\eta = \theta/\alpha$ and $u_r = g(\eta)u_{max}$ where u_{max} depends only on the radius r (the result is based on similarity solutions).

Knowing the analytical solution, the analysis of the Jeffery-Hamel problem is straightforward. Carrying out the calculations, one can claim that the Jeffery-Hamel problem is not an analytical solution of the LBM. Both the density and velocity errors are function of several parameters: x, y, α, ρ, and u_{max}. In spite of this fact, following observations can be given, using, e.g., $x=1$, $y=0$, $\alpha=\pi/4$, and $\rho=1$:

(1) The compressibility error is $O(M^2)$.

(2) The compressibility error is sixth order in space, but at a certain Mach number it starts to decrease. Keeping in mind that this is a creeping flow and on increasing u_{max}, the solution still remained second order for relatively large Mach numbers.

(3) The convergence of the velocity is second order in space (the rate of convergence was 2.3), in the region where the Mach number is small enough to give sixth order accuracy in space.

(4) We observed similar behavior at other space positions (x,y) although the rate of convergence changed a little (e.g., at $x=2$, $y=0.5$ the rate of convergence decreased to 2).

All of these results suggest that there should be some general relation between the compressibility error and the velocity field, and this relation can be simple enough for low orders in space.

V. ANALYTICAL SOLUTIONS FOR THE LATTICE BOLTZMANN METHOD

Using this technique, one can look for a general class of analytical solutions of the LBM. It is important to note that some of these solutions can be analytical solutions of the Navier-Stokes equations, too. First, we limit the scope of such analysis to 2D and more restrictively for solutions, which can be given as $u_x = u_x(y)$ and $u_y = 0$. Let us align the lattice main diagonal parallel to the flow field. We know that the Poiseuille profile provides a specific analytical solution for this situation. Substituting the general solution to the equilibrium distributions and performing the analysis, one can obtain some interesting results. The density is always analytical, i.e., there is no compressibility error at all for this type of flows. The velocity error obtained for the x component can be written as

$$u_{x,err} = -1/6[2u_x(y) - 6G_x\delta/\rho - u_x(y^-) - u_x(y^+)],$$
$$u_{y,err} = G_y/\rho. \qquad (16)$$

It is worth emphasizing that the same expressions for the error can be derived for the seven velocities (D2Q7) model. To eliminate these errors, one has to select u_x and a corresponding body force. Obviously, if there is no y component of the body force, i.e., $G_y = 0$ then u_y is also analytical. It is straightforward to prove that a possible choice with first-order y is

$$u_x = u_0 y, \quad u_y = 0, \quad G_x = G_y = 0, \qquad (17)$$

which is the Couette flow. Actually, it has already been pointed out in another way [6] that this is also an analytical solution of the LBM. A second-order solution can be given in the form of the Poiseuille profile and the corresponding body force given by Eq. (4). It is easy to show that a third-order solution also exists, which—as far as we know—has not been known before:

$$u_x = u_0(1 - y^3), \quad u_y = 0, \quad G_x = -6\rho\nu y u_0. \qquad (18)$$

In fact, one can point out that there are infinite number of analytical solutions for the LBM and incorporating Eq. (16) with the corresponding Navier-Stokes equations, one can derive a simple ordinary differential equation to obtain solutions that verify both the lattice Boltzmann and the Navier-Stokes equations [12].

VI. ON THE COMPRESSIBILITY ERROR

A nice property of this analysis is that it can provide information about the compressibility error independently from the discretization error. Indeed, using the same technique as before, one can derive the compressibility error for a general problem where $u_x \equiv u_x(x,y)$ and $u_y \equiv u_y(x,y)$. At a given lattice site, the resulting error term is a function of the velocities of the neighboring sites. One can take the Taylor expansion up to second-order of these velocities and using the fact that the velocity field should be divergence free for incompressible flow, one can simplify the error term as follows:

$$\rho_{err} = \delta^2\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \frac{\partial u_y}{\partial x}\frac{\partial u_x}{\partial y}\right]. \qquad (19)$$

The compressibility error is clearly second order and Eq. (19) gives a quite simple relation between the velocity field and the compressibility error. Obviously, this error term disappears (just like higher-order terms) for unidirectional flows, when the main diagonal is parallel with the flow field.

One can also check that the second-order error, Eq. (19), will be zero for the rotated flow field [see, e.g., Eq. (6)], too.

VII. CONCLUSION

The accuracy of the LBM may change with the orientation of the flow field and the lattice. It is worth emphasizing that the results here do not contradict earlier observations. Although the accuracy is changing with the lattice orientation for the Poiseuille problem, the worst case still gives third-order accuracy in space for the velocity, whereas the multiscale expansion predicts only second-order accuracy for the LBM. Deviations from incompressible behavior are also known as $O(M^2)$ [13], which is also in line with the results obtained here [see Eq. (10)].

Our analysis proved that the Jeffery-Hamel flow is not an analytical solution of the LBM, however its analysis provided interesting observations about the accuracy of the LBM. We have already called the reader attention to the fact that in spite of the lack of analytical solution, one can use the accuracy information obtained from analytical test to *a priori* investigate the effect of boundary conditions on the accuracy.

The calculation introduced in this paper is straightforward using modern symbolic manipulation tools. We gave a simple condition for a specific class of hydrodynamic problem, which—in different orders—resulted in, subsequently, the Couette, the Poiseuille flow, and in third-order a new accurate solution for the LBM. All of these solutions verify the 2D Navier-Stokes equations, which, we believe, make such an analysis really exciting, especially since the analysis can be extended easily for 3D.

Finally, it was pointed out that the compressibility error is in a strong relation with the velocity field. More details about the nature of this error will be presented in the near future.

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